MATHEMATICS

Duration: Three Hours *Maximum Marks: 150*

- **Read the following instructions carefully**
  1. Write all the answers in the answer book.
  2. This question paper consists of TWO SECTIONS: A and B.
  3. Section A has Eight questions. Answer ALL questions in this section.
  4. Section B has Twenty two questions. Answer any TEN questions in this section. Only first ten un scored answers will be considered. Score off the answers which are not to be evaluated.
  5. Answers to Section B should start on a fresh page and should not be mixed with answers to Section A.
  6. Answers to questions and answers to parts of a question should appear together and should not be separated.
  7. In all questions of 5 marks, write clearly the important steps in your answer. These steps carry partial credit.
  8. There will be no negative marking.
- **Note**

  The symbols N, Z, Q, R, C, M_n and $C'([0,1],R)$ denote, respectively, the set of natural numbers integers, rational numbers, complex numbers, $n\times n$ matrices with real entries and all continuously differentiable real valued functions defined on $[0,1]$.

SECTION – A (100 Marks)

1. This question has 20 parts. Answer all parts. Each part carries 2 marks. For each part only one of the suggested alternatives is correct. Write the alphabet corresponding to the correct alternative in your answer book. (20 × 2 = 40)

1.1. Let A be a $m\times n$ matrix with row rank r = column rank. The dimension of the space of solutions of the system of linear equations AX = 0 is

(A) r

(B) $n-r$

(C) $m-r$

(D) $\min.(m,n)-r$

1.2 A matrix M has eigen values 1 and 4 with corresponding eigen vectors $(1,-1)^T$ and $(2,1)^T$, respectively. Then M is

(A) $\begin{pmatrix}-4 & -8\\ 5 & 9\end{pmatrix}$

(B) $\begin{pmatrix}9 & -8\\ 5 & -4\end{pmatrix}$

(C) $\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$

(D) $\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}$

1.3 Let PID, ED, UFD denote the set of all principal ideal domains, Eucliden domains, unique factorization domains, respectively. Then

(A) $UFD\subset ED\subset PID$

(B) $PID\subset ED\subset UFD$

(C) $ED\subset PID\subset UFD$

(D) $PID\subset UFD\subset ED$

1.4 Let $f:C\to C$ be given by

$$f(z)=\begin{cases}(\bar{Z})^2/Z & \text{when } Z\neq 0\\ 0 & \text{when } Z=0\end{cases}$$

Then f

(A) is not continuous at Z = 0

(B) is differentiable but not analytic at Z=0

(C) is analytic at Z = 0

(D) satisfies the Cauchy–Riemann equations at Z = 0

1.5 The bilinear transformation $w=2Z(Z-2)$ maps $\{Z:|Z-1|<1\}$ onto

(A) $\{w:\operatorname{Re} w<0\}$

(B) $\{w:\operatorname{Im} w>0\}$

(C) $\{w:\operatorname{Re} w>0\}$

(D) $\{w:|w+2|<1\}$

1.6 The statement "The dual space of a nonempty nor med linear space is non empty" follows from

(A) Uniform bounded ness principle

(B) Hahn-Banach theorem

(C) Riesz representation theorem

(D) Closed graph theorem

1.7 Let $T : D \subset X \to Y$ be a linear transformation, where X and Y are nor med linear spaces. Then T is closed if and only if

(A) D is closed

(B) T (D) is closed

(C) Graph of T is closed

(D) T(D) is bounded

1.8 The Hermite interpolating polynomial for the function $f(x) = x^6$ based on –1, 0 and 1 is

(A) $x^4 - 2x^2$

(B) $2x^4 - x^2$

(C) $x^2 + 2x^2$

(D) $2x^4 + x$

1.9 The system if equations

$3x + 2y = 4.5$

$2x + 3y - z = 5.0$

$-y + 2z = -0.5$

Is to be solved by successive over relaxation method. The optimal relaxation factor. $\omega_{opt'}$ rounded up to two decimal places is given by

(A) 1.23

(B) 0.78

(C) 1.56

(D) 0.63

1.10 The Fourier series of the $2\pi -$periodic function $f(x) = x + x^2, -x < x \le \pi$ at $x = \pi$ converges to

(A) π

(B) 2π

(C) π^2

(D) $\pi + \pi^2$

1.11 Let S be a non empty Lebesgue measurable subset of R such that every subset of S is measurable. Then the measure of S is equal to the measure of any

(A) Subset of S

(B) Countable subset of S

(C) Bounded subset of S

(D) Closed subset of S

1.12 If $y' - x \neq 0$, a solution of the differential equation $y'(y' + y) = x(x + y)$ is given by y =

(A) $1 - x - e^{-x}$

(B) $1 - x + e^{x}$

(C) $1 + x + e^{-x}$

(D) $1 + x + e^{x}$

1.13 For the differential equation $4x^3y'' + 6x^2y' + y = 0$, the point at infinity is

(A) An ordinary point

(B) A regular singular point

(C) An irregular singular point

(D) A critical point

1.14 Using the transformation $u = \frac{w}{y}$ in the partial differential equation $xu_x = u + yu_y$, the transformed equation has a solution of the form w =

(A) $f\frac{x}{y}$

(B) $f(x + y)$

(C) $f(x - y)$

(D) $f(xy)$

1.15 The complete integral of the partial differential equation $xp^3q^2 + yp^2q^3 + (p^3 + q^3) - Zp^2q^2 = 0$ is Z =

(A) $ax + by + (ab^{-2} + ba^{-2})$

(B) $ax - by + (ab^{-2} - ba^{-2})$

(C) $-ax + by + (ba^{-2} - ab^{-2})$

(D) $ax+by-\left(ab^{-2}+ba^{-2}\right)$

1.16 Introducing the new coordinate $Q=1\sin\omega t$ in the Lagrangian

$L=\frac{m}{2}\left(q^2\sin^2\omega t+qq\omega\sin 2\omega t+q^2\omega^2\right)$ the Hamiltonian H equals (p is conjugate momentum)

(A) $\frac{m}{2}\left(\frac{p^2}{m^2}-Q^2\omega^2\right)$

(B) $\frac{m}{2}\left(\frac{p^2}{m^2}+Q^2\omega^2\right)$

(C) $\frac{m}{2}\left(\frac{p^2}{m^2}-Q^2\omega^2+2\omega^2Q\operatorname{cosec}\omega t\right)$

(D) $\frac{m}{2}\left(\frac{p^2}{m^2}+Q^2\omega^2-2\omega^2Q\operatorname{cosec}\omega t\right)$

1.17 As a subset of $[0,1]$ equipped with the usual topology, the cantour set is

(A) Closed but not compact, nowhere dense and uncountable

(B) Not closed, dense and countable

(C) Closed, dense and un countable

(D) Compact, nowhere dense and uncountable

1.18 If A and B are two events and the probability $P(B)\neq 1$, then

$\frac{P(A)-P(A\cap B)}{1-P(B)}$ equals

(A) $P\left(A/\bar{B}\right)$

(B) $P(A/B)$

(C) $P\left(\bar{A}/B\right)$

(D) $P\left(\bar{A}/\bar{B}\right)$

1.19 Let A, B and C be three independent events. If

α = Pr(A occurs),

β = Pr(not all three events occur simultaneously)

γ = Pr(at least one of the three events does not occurs),

δ = Pr(C occurs, but neither A nor B occur),

Then the probability that C occurs equals

(A) $\delta/(\beta+\delta)$

(B) $\beta/(\beta+\delta)$

(C) $\alpha\gamma(\beta+\delta)$

(D) $\gamma/(\beta+\delta)$

1.20 A degenerate basic feasible solution of the convex region formed by the following closed half spaces in R^2:

$x_1+3x_2\le 12$

$x_1+x_2\le 6$

$2x_1-x_2\le 6$

$0\le x_1\le 4, x_2\ge 0$

(A) (3,0)

(B) (0,4)

(C) (4,2)

(D) (3,3)

2. This question has 10 parts. Answer all parts. Each part carries 3 marks. The answers expected in the blanks should be written in your answer book. (10 × 3 = 30)

2.1 If $\theta\in R\setminus\{n\pi:n\in Z\}$ and P is a 2 × 2 matrix with complex entries such that

$$P^{-1}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.P=\begin{pmatrix}e^{i0} & 0\\ 0 & e^{-i0}\end{pmatrix}$$

then P = ______________.

2.2 The order of the group generated by the matrices $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$, where $i=\sqrt{-1}$, under matrix multiplication is ____________.

2.3 The complex analytic function, $f(z)$, with the imaginary part $e^x(y\cos y+x\sin y)$ is ______________.

2.4 The range of the absolute stability of the multi step method

$$y_{n-1}=y_n+\frac{h}{12}\left(23f_n-16f_{n-1}+5f_{n-2}\right)$$

For the differential equation $y'=f(x,y)$ is ______________.

2.5 The line integral of $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ on the circle $x^2 + y^2 = a^2, z = 0$ described in the clockwise sense is ______________.

2.6 The Green's function for the boundary value problem

$y'' + f(x) = 0,\ y(0) = 1,\ y(\pi) = -1,$

is ______________.

2.7 The characteristics of the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \cos^2 x \frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 0$$

When it is of hyperbolic type are _____________ and _____________.

2.8 For the Hamiltonian $H = \frac{1}{2}\left(q^2 + p^2 q^4\right)$ the equation of the motion for q can be written as $f(q, \dot{q}, \ddot{q}) =$ ___________.

2.9 If we identity points $(x,0)$ with $(2\pi - x, 2\pi)$ in $[0,2\pi]\times[0,2\pi]$ equipped with the product topology, the identification space under this equivalence is ___________________.

2.10 The quadratic equation $x^2 - ax + b = 0$ is known to have two real roots, x_1, and $x_2\ (x_1 > x_2)$, but the coefficient b is a positive unknown and can be assumed to have a uniform distribution in the permissible range of variation. The expected value of x_1 is ________________.

3. Let T be a linear operator defined on a finite dimensional vector space V. If rank $(T^2) =$ rank (T), find $R(T) \cap N(T_2)$ where $R(T), N(T)$ denote respectively the range and the null space of T. (5)

4. Using the Residule Theorem and the contour shown, evaluate $\int_0^\infty \frac{dx}{1+x^3}$. (5)

5. Determine a, b and c such that the iterative method $x_{n+1} a x_n^4 + b x_n^4 + x_n, c,\ n \ge 0,$ converges to 2 with cubic convergence. Taking the itial approximation $x_0 = 1.5$, determine the smallest integer n for which $|x_n - 2| < 10^{-6}$ holds. (5)

6. Define $\{a_n(x)\}_{n\ge 0}$ on $[-1,1]$ as

$$a_{n+1}(x) = a_n(x) + \frac{1}{2}\left[x^2 - \{a_n(x)\}^2\right],\ n \ge 0,$$

Show that $\{a_n(x)\}$ converges uniformly to $|x|$ on $[-1,1]$. (5)

7. Let $f_1, f_2, \ldots\ldots f_n$ be real valued functions defined on an interval $[a,b]$ such that each f_i has continuous derivatives up to order $(n-1)$. If the Wronskin $W(f_1, \ldots\ldots, f_n)(x) = 0$ and $W(f_1, \ldots., f_{n-1})(x) \ne 0$ in $[a,b]$, Show that $f_1, f_2, \ldots., f_n$ are linearly dependent on $[a,b]$. (5)

8. Find the characteristic function of $Y = \sum_{r=1}^{n} a_r X_r,$ where $a_1, a_2, \ldots\ldots, a_n$ are constants and $X_2, X_2, \ldots\ldots, X_n$ are independent random variables, each of which takes the values –1 and 1 with probability $\frac{1}{2}$. Taking $a_r = 2^{-r}$ for each r, show that Y converges in distribution to uniform distribution on $(-1,1)$.

SECTION – B (50 Marks)

9. Let A be a 6 × 6 diagonal matrix with characteristic polynomial $x(x+1)^2(x-1)^3$. Find the dimension of γ, where $\gamma = \{B \in M_6(R) : BA = AB\}$.

10. Show the every cyclic group of order n has a unique subgroup of order d, for each $\frac{d}{n}$. Deduce that $\sum_{d/n} \phi(d) = n,$ where ϕ is the Euler phi function.

11. If $A_1, A_2, \ldots\ldots\ldots, A_n$ are ideals in a ring R such $A_i + A_j = R\forall i$. Show that

$R = A_1 + (A_2 \cap A_3 \cap ... \cap A_n)$. Using this, show that

$$\frac{R}{\bigcap_{i=1}^{n} A_i} \cong R/A_1 \times R/A_2 \times ...... \times R/A_n$$

12. Show that $\oint_{z-a/=R} p(z)\,d\bar{z} = -2zi\,R^2 p'$ (a) , where $p(z)$ is a polynomial and $a \in C$.

13. Let f be an entire function such that $|f(Z)| \le |Z|^C$ for $|Z| \ge A$, for some positive constants A and C. Show that f is a polynomial of degree at most C.

14. Let A be a closed subspace of the space $C'([0,1]R)$ and g be a real valued function on $[0,1]$ such that $gf \in A \forall f \in A$. Show that the linear transformation $M : A \to A$ given by $M(f) = gf$ is continuous. Where $(gf)(x) = g(x) f(x) \forall x \in R$.

15. Let M be a closed subspace of a Hilbert space H. For $x_0 \in H$, show that

$$\min\{\|x - x_0\| : x \in M\} = \max\{|\langle x_0, y\rangle| : y \in M^1, \|y\|\} = 1$$

16. Using the multi step method $y_{n+2} = y_n + \frac{h}{3}(y_n' + 4y_{n+1}' + y_{n+2}')$ compute $y(0.6)$ for the differential equation $y' = x(y+x)$ with $h = 0.2, y(0) = 1$ and $y(0.2) = 1.1$.

17. Determine W_0, W_1 and W_2 as functions of α such that the error R in

$$\int_{-1}^{1} f(x)\,dx = W_0 f(-\alpha) + W_1 f(0) + W_2 f(\alpha) + R,$$

$\alpha \ne 0$

vanishes when $f(x)$ is an arbitrary polynomial of degree at most 3. Show that the precision is five when $\alpha = \sqrt{3/5}$ and three otherwise. Compute the error R when $\alpha = \sqrt{3/5}$.

18. For the vector field $\vec{F} = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} + \sin^3 y \cos^2 z\vec{k}$, evaluate $\oint_0 \vec{F}.d\vec{r}$, where C is the closed contour in the xy – plane consisting of the parabolas $y = \pm(x^2+1)$ and the straight lines $x = \pm 1$.

19. Let $f:[a,b] \to R$ be absolutely continuous, and let A be a subset of $[a,b]$ of measure 0. Show that the measure of $f(A)$ equals 0.

20. If $(az+by^2)dx + (cz+ex^2)dy$ is an exact differential in x and y, show that

$$z = 2bxy + \left(\frac{ab}{c} - e\right)x^2 + f(ax+cy)$$

Where a, b, c and e are non zero constants and f is an arbitrary function.

21. Let φ_i be real valued nontrivial solution of $y'' + a_j(x)y = 0$ on (a,b) for $i = 1,2$. If $a_2(x) > a_1(x)$ on (a,b), show that between any two zeros of φ_1 there exists a zero of φ_2. Hence show that a solution of $y'' + (1+\sin x)y = 0$ has at least one zero in each of the intervals $[2n\pi, (2n+1)\pi], n \in Z$.

22. Let $u_1(x,y)$ and $u_2(x,y)$ be the solutions to the Cauchy problems $u_{xx} + u_{yy} = 0, u(x,0) = f(x)$, $u_y(x,0) = g(x) + h(x)$, where f and g are differentiable everywhere and

$$h(x) = \begin{cases} 0 & \text{If } u = u_1 \\ \frac{1}{n}\sin nx & \text{If } u = u_2, n \in N \end{cases}$$

Show that

$$u_2(x,y) - u_1(x,y) = \frac{1}{n^2}\sin h(ny)\sin(nx)$$

Draw conclusions about the continuous dependence of the solutions on the initial values.

23. The ends A and B of a rod 20 cm in length are kept at temperatures 30°C and 80°C until the steady state condition prevails. Suddenly the temperature at the end A is increased to 40°C and at the end B is decreased to 60°C. Find the temperature distribution in the rod at time t.

24. A satellite of mass m is orbiting the earth at some distance from it. Assuming that the inverse square law for the force towards the centre of the earth holds and that the path is planar, show that the Lagrangian L can be written as

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GmM}{r},$$

Where $r.\theta$ are plane polar coordinates., Deduce that $r^2\theta = h$(constant) and obtain the Lagrange equation in the coordinate r. Prove that the total energy is conserved

25. (a) A uniform beam of length 32m rests in equilibrium against a smooth wall and upon a smooth peg at a distance 2m from the wall. Using the principal of virtual work, show that in this position of equilibrium the beam is inclined to the wall at an angle of 30°.

(b) A uniform rod of length $2l$ rsts in a horizontal position on a fixed horizontal cylinder of radius a. It is displaced in a vertical plane and rock without slipping. If θ be its angle of inclination with the horizontal, then the equation of motion can be written as

$$\left(\frac{l^2}{3} + a^2\theta^2\right)\dot{\theta}^2 + 2ga\left(\cos\theta + \theta\sin\theta\right) =$$

constant.

If the oscillation is small, show that the time of oscillation is $2\pi\sqrt{l^2/3ga}$.

26. Define a topology τ^* on $X = [0,1] \subset N$ by $\tau^* = \tau \cup \{[0,1] \cup A : A \subseteq N\}$, where τ is the usual topology on $[0,1]$. Explain whether or not:

a. (X, τ^*) is Hausdorff,

b. (X, τ^*) is compact,

c. $[0,1]$ is a compact subset of (X, τ^*)

d. $[0,1]$ is dense in (X, τ^*)

27. Suppose (X, τ) is a T_4-space in which every closed set is a $G_\delta-$set. Show that for each pair of disjoint closed sets A and B in X, there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

28. Let p be the observed proportion of successes in a sample of size n. Find the confidence limits for estimating the population proportion of successes $\hat{p}$ at α level of confidence determined by Z_α and show that for large n, it can be approximated by $\hat{p} \approx p \pm Z_\alpha\sqrt{\frac{p(1-p)}{n}}$.

29. It is desired to test whether two methods of learning are equally good. Two groups of students, A and B, are made to learn the same material by the two methods. Assume that the two methods really do give different results with error variances σ_1^2 and σ_2^2, respectively. Further assume that the cost of making an observation in the groups A and B is the ratio of $1:\alpha$, where $1 > \alpha$. If the total number of students in the two groups is fixed, show that the most precise method of estimating the difference in the methods is to allocate the number of students in the two groups in the ratio of $\sigma_1\sqrt{\alpha} : \sigma_2$.

30. Find the range for $Q \le R$ for which the basis corresponding to the feasible solution (3,4) for the problem P(0) remains optimal for the problem P(Q), where P(Q) is

$\max : 3x_1 + 5x_2$

Subject to $x_1 + 3x_2 \le 15 - 2Q$

$x_1 + x_2 \le 7 + Q$

$2x_1 - x_2 \le 8 - 3Q$

$x_1, x_2 \ge 0$.