MATHEMATICS

Duration: Three Hours *Maximum Marks: 150*

- **Read the following instructions carefully**
  1. All answers must be written only in the answer book provided.
  2. This question paper consists of TWO SECCTIONS: A and B.
  3. Section A consists of two questions of the multiple choice type. Question 1 consists of THIRTY FIVE sub-questions of TWO marks each and Question 2 consists of FIVE sub-questions of ONE mark each.
  4. The answers to multiple choice questions must be written only in the boxes provided in the first two sheets of the answer book.
  5. Answers to Section B should be start on a fresh page and should not be mixed with answers to Section A. Question numbers must be written legibly and correctly in the answer book.
  6. Section B consists of TWENTY questions of FIVE marks each ANY FIFTEEN out of them have to be answered. If more number of questions are attempted, score off the answers not to be evaluated, else only the first fifteen un scored answers will be considered strictly.
  7. In all 5 mark questions, clearly show that the important steps in your answers, these steps will carry partial credit.
  8. There will be NO NEGATIVE marking.
- **Note**

  The symbols R and C respectively denote the set of all real numbers and complex numbers. Vector quantities are denoted by bold letters.

SECTION A (75 Marks)

1. This question consists of 35 (thirty five) sub-questions, each carrying two marks. Answer all the sub-questions. For each sub-question only one of the suggested alternatives is correct. The alphabet corresponding to the correct alternative MUST be written only in the boxes corresponding to the questions in the first sheet of the answer book.

1.1 Let P be a matrix of order m × n and Q be a matrix of order n × p, n ≠ p. If rank (P) = n and rank (Q) = p, then rank (PQ) is

(A) n

(B) p

(C) np

(D) n+p

1.2 Let P and Q be square matrices such that PQ = I, the identity matrix. Then zero is an eigen value of

(A) P but not of Q

(B) Q but not of P

(C) Both P and Q

(D) Neither P nor Q

1.3 An analytic function $f(z)$ is such that $\operatorname{Re}\{f'(z)\}=2y$ and $f(1+i)=2$. Then the imaginary part of $f(z)$ is

(A) $-2xy$

(B) x^2-y^2

(C) $2xy$

(D) y^2-x^2

1.4 The value of the integral $\oint_C \frac{\sin \pi z^2+\cos \pi z^2}{(x-4)(z-2)}dz$. Where C is the circle $|z|=3$ traced anti-clockwise, is

(A) $-2i\pi$

(B) $i\pi$

(C) $-i\pi$

(D) $2i\pi$

1.5 For the function $f(z)=\frac{z-\sin z}{z^3}$, the point z = 0 is

(A) A pole of order 3

(B) A pole of order 2

(C) An essential singularity

(D) A removable singularity

1.6 Let E be the set of all number x in [0,1] such that the decimal expansion of x does not contain the digit 7. Then the Lebesgue measure of E is

(A) 0

(B) 0.7

(C) 0.9

(D) 1

1.7 Let $f_n : [0,1] \to R$ be defined by

$f_n(x) = 2n,$ if $\frac{2}{2n} \le x \le \frac{1}{n}$

= 0 otherwise

Then the value of $\int_0^1 \lim_{n\to\infty} f_n \, d\mu$ and $\lim_{n\to\infty} \int_0^1 f_n \, d\mu$ (where μ is the Lebesgue measure on R) are respectively

(A) 0,0

(B) 0,1

(C) 1,0

(D) 1,1

1.8 The value of the surface integral

$\iint_s F.n \, dS,$

Where S is the surface of the sphere $x^2 + y^2 + z^2 = 4,$

n is the unit outward normal and $F = xl + yj + zk,$ is

(A) 32π

(B) 16π

(C) 8π

(D) 64π

1.9 The value of the line integral

$\int_L y\,z\,dx + x\,z\,dy + x\,y\,dz$

Where L is the line segment from $(1,1,0)$ to $(2,3,2)$ is

(A) 0

(B) 9

(C) 7

(D) 12

1.10 Let $f(x,y) = (x-2)^2 (y+3)$. Then

(A) $(2,-3)$ is not a stationary point of f

(B) f has a local maximum at $(2,-3)$

(C) f has a local minimum at $(2,-3)$

(D) f has neither a local maximum nor a local minimum at $(2,-3)$

1.11 The differentiable equation

$(3a^2x^2 + b\,y\cos x)\,dx + (2\sin x - 4ay^3)\,dy = 0$

is exact for

(A) $a = 3, b = 2$

(B) $a = 2, b = 3$

(C) $a = 3, b = 4$

(D) $a = 2,\ b = 5$

1.12 The eigen values for the boundary value problem

$x'' + \lambda x = 0$; $x(0) = 0,\ x(\pi) + x'(\pi) = 0$,

satisfy

(A) $\lambda + \tan \lambda\pi = 0$

(B) $\sqrt{\lambda} + \tan \lambda\pi = 0$

(C) $\sqrt{\lambda} + \tan \sqrt{\lambda}\ \pi = 0$

(D) $\lambda + \tan \sqrt{\lambda}\ \pi = 0$

1.13 If $2x(1-y) = K$ and $g(x,y) = L$ are orthogonal families of curves where K and L are constants, then $g(x,y)$ is

(A) $x + 2y - y^2$

(B) $2y(1-x)$

(C) $x^2 + 2x - y^2$

(D) $x^2 + 2y + y^2$

1.14 Let S_3 be the group of all permutations on three symbols, with the identity element e. Then the number of elements in S_3 that satisfy the equation $x^2 = e$ is

(A) 1

(B) 2

(C) 3

(D) 4

1.15 Let G be a group of order 15. Then the number of Sylow subgroups of G order 3 is

(A) 0

(B) 1

(C) 3

(D) 5

1.16 For $j = 1, 2, ...,$ let $e_j = (a_1, a_2, a_3 '........)$

Where $a_i = 1$ and $a_i = 0$ for $i \neq j$.

Then $E = \{e_1, e_2, e_3, ...\}$ is a

(A) Hamel basis of l^1

(B) Hamel basis of l^∞

(C) Schauder basis of l^1

(D) Schauder basis of l^∞

1.17 Let X be the space of all real valued continuously differentiable functions of [0,1] and Y be the space of all real valued continuous functions on [0,1], both with supremum norm. Let $T : X \to Y$ be defined by

$$(Tx)(t) = x'(t) + \int_0^t x(s)\,ds,\ 0 \le 1 \le 1,\ x \varepsilon X.$$

Then T is

(A) Continuous and the graph of T is not closed

(B) Not continuous and the graph of T is closed

(C) Continuous and the graph of T is closed

(D) Not continuous and the graph of T is not closed.

1.18 The maximum step size h such that the error in linear interpolation for the function $y = \sin x$ in $[0, \pi]$ is less than 5×10^{-5} is

(A) 0.02

(B) 0.002

(C) 0.04

(D) 0.06

1.19 The second order Runge-Kutta method is applied to the initial value problem $y' = -y,\ y(0) = y_0$, with step size h. Then y(h(is

(A) $y_0 (h-1)^2$

(B) $\frac{y_0}{2}(h^2 - 2h + 2)$

(C) $\frac{y_0}{6}(h^2 - 2h + 2)$

(D) $y_c\left(1 - h + \frac{h^2}{2} + \frac{h^3}{6}\right)$

1.20 The interative scheme $x_{n+1} = \frac{X_n}{2}\left(1 + \frac{a}{x_n^2}\right)$ converges to $\sqrt{a}$. The convergence is

(A) Linear

(B) Quadratic

(C) Cubic

(D) Biquadratic

1.21 The value of the integral $\int_{-\pi}^{\pi} \frac{dt}{4 + r^2 - 4r\cos(\theta - t)}, -\pi < 0 \le \pi,\ 0 < r < 2$ is

(A) 0

(B) $2\pi r^2 (\cos\theta - \sin\theta)$

(C) $\frac{4 - r^2}{2\pi}$

(D) $\frac{2\pi}{4 - r^2}$

1.22 Let $u(x,t)$ be the solution of

$u_n = u_{xx}; 0 < x < 1,\ t > 0,\ u(x,0) =$
$x(1-x),\ u(x,0) = 0$

Then $u\left(\frac{1}{2}, \frac{1}{4}\right)$ is

(A) $\frac{3}{16}$

(B) $\frac{1}{4}$

(C) $\frac{3}{4}$

(D) $\frac{1}{16}$

1.23 Under the influence of a force field F, a particle of mass 'm' moves along the ellipse $r = a\cos\omega t\, i + b\sin\omega t\, j$. Then $r \times F$ is

(A) 0

(B) $m\omega^2\left(a^2\cos^2\omega t + b^2\sin^2\omega t\right)k$

(C) $m\omega^2\left(a^2 + b^2\right)\cos\omega t\sin\omega t k$

(D) $m a^2\omega^2 k$

1.24 A particle of mass 4 units moves along x–axis attracted towards the origin by a force whose magnitude is 8x. If it is initially at rest at x = 10, then the frequency of the particle is

(A) $10\sqrt{2}\pi$

(B) π

(C) $\frac{\pi}{10\sqrt{2}}$

(D) $\sqrt{2}\pi$

1.25 The Hamiltonian of a system is given by

$$H\left(q_1, q_2; p_1, p_2\right) = kp_1^2 + \frac{k}{q_1^2}p_2^2 + \frac{1}{q_1}$$

Where q_1, q_2 are the generalized coordinates, p_1, p_2 are the generalized momenta and k, *l* are constants. Then

(A) $p_1 = kt + l$

(B) $p_2 = kt + l$

(C) p_1 is independent of time

(D) p_2 is independent of time

1.26 Let $X = [0,1] \cup [2,3]$ with the subspace topology induced by the usual topology on R and let $f : X \to R$ be defined by

$f(x) = x$ if $0 \le x < 1$

$= 2$ if $2 \le x \le 3$

Then

(A) $f(G)$ is open in R for every open set G in X

(B) $f(H)$ is closed in R for every closed set H in X

(C) f is a continuous function

(D) f is a discontinuous function

1.27 Let X = R with confinite topology. Then X is a

(A) First countable space

(B) T_1 – space

(C) Regular space

(D) Normal space

1.28 Two independent events E and F are such that $P(E \cap F) = \frac{1}{6}$, $P(E^c \cap F^c) = \frac{1}{3}$ and $P(E) > P(F)$. Then P(E) is

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{1}{3}$

(D) $\frac{1}{4}$

1.29 A box contains two coins, one of which is fair and the other two headed. One coin is chosen at random and tossed twice. If two heads appear, then the probability that the chosen coin was two headed is

(A) $\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{1}{3}$

(D) $\frac{4}{5}$

1.30 A random variable X has passion distribution. If

$2P(X=2)=P(X=1)+2P(X=0)$, then the variance of X is

(A) $\frac{3}{2}$

(B) 2

(C) 1

(D) $\frac{1}{2}$

1.31 The joint probability density function of the random variables X, Y and Z is

$f(x,y,z)=8xyz, 0<x,y,z<1$

= 0 otherwise

Then $P(X<Y<Z)$ is

(A) $\frac{1}{8}$

(B) $\frac{1}{3}$

(C) $\frac{1}{6}$

(D) $\frac{3}{8}$

1.32 Consider the following primal problem:

Minimize $z=10x_1+x_2+5x_3$

Subject to $5x_1-7x_2+3x_3 \geq 50,$

$x_1, x_2, x_3 \geq 0$

The optimal value of the primal is

(A) $\frac{50}{3}$

(B) $\frac{10}{3}$

(C) $\frac{250}{3}$

(D) $\frac{100}{3}$

1.33 Consider the linear programming problem: Maximize $z=2x_1-4x_2$ subject to $x_1+2x_2 \leq 3,\quad 3x_1+4x_2 \leq 5,\ x_1,x_2 \geq 0$. The total number of basic solutions is

(A) 6

(B) 2

(C) 4

(D) Infinity

1.34 The solution of the integral equation

$g(s)=s+\int_0^1 su^2 g(u)du$ is given by

(A) $g(t)=\frac{3t}{4}$

(B) $g(t)=\frac{4t}{3}$

(C) $g(t)=\frac{2t}{3}$

(D) $g(t)=\frac{3t}{2}$

1.35 The function $f(x)=e^x$ has the Fourier expansion $e^x=\sum_1^{\infty} b_n \sin(nx)$ in the interval $(0,\pi)$. Then $\sum_1^{\infty}(b_n)^2$ converges to

(A) $\frac{1}{\pi}(e^{\pi}-1)$

(B) $\frac{1}{\pi}(e^{\pi}+1)$

(C) $\frac{1}{\pi}(e^{2\pi}-1)$

(D) $\frac{1}{\pi}(e^{2\pi}+1)$

2. This equation consists of 5 (Five) sub-questions, each one mark. Answer all the sub-questions. For each sub-question, only one of the suggested alternatives is correct. The alphabet corresponding to the correct alternative MUST be written only in the boxes corresponding to the questions in the second sheet of the answer book.

2.1 Let W be an m-dimensional subspace of an n-dimensional vector space V, where $m<n$. Then the dimension of V/W is

(A) $\frac{n}{m}$

(B) $n-m$

(C) $n+m$

(D) 0

2.2 Each of the following subsets $\Phi, R, (0,1), [0,1]$ of R, with the usual metric, is

(A) Complete

(B) Compact

(C) Connected

(D) Bounded

2.3 The value of the Wronskian of the functions $x^2, 3x+2$ and $2x+3$ is

(A) 0

(B) –10

(C) –5

(D) 8

2.4 Let Z_{10} denote the ring of integers modulo 10. Then the number of ideals in Z_{10} is

(A) 2

(B) 3

(C) 4

(D) 5

2.5 A particle is placed on the top of a sphere in a gravitational field and allowed to slide without friction. Then the motion has

(A) No constraint

(B) A holonomic constraint

(C) A non-holonomic constraint

(D) A rheonomic constraint

SECTION B (75 Marks)

This section consists of 20 (Twenty) questions of 5 (Five) marks each. ANY 15 (Fifteen) of them have to be answered. If more number of questions are attempted, score off the answers not to be evaluated, else only the first fifteen un scored answers will be considered.

3. Let V and W be finite dimensional vector spaces. Let $T: V \to W$ be a linear transformation and $\{u_1, u_2, ... u_n\}$ be a subset of V such that $\{Tu_1, Tu_2, ... Tu_n\}$ is linearly independent in W. Then show that $\{u_1, u_2, ... u_n\}$ is linearly independent in V. Deduce that, if T is onto, then $\dim V \geq \dim W$.

4. Let V be an inner product space over R and $T: V \to V$ be a linear transformation such that $\langle Tu, v \rangle = \langle u, Tv \rangle$ and $\langle Tu, u \rangle \geq 0$ for all $u, v \in V$. Prove that

$$\left| \langle Tu, v \rangle \right|^2 \leq \langle Tu, u \rangle \langle Tv, v \rangle \text{ for all } u, v \in V.$$

5. Evaluate $\int_0^{2\pi} \cos^{2n} \theta d\theta$ using contour integration.

6. Show that every bilinear transformation $w = f(z)$ having fixed points a and b is given by

$$\frac{(w-a)}{(w-b)} = K \frac{(z-a)}{(z-b)} \text{ for } a \neq b$$

And

$$\frac{1}{(w-a)} = \frac{1}{(z-a)} + 1 \text{ for } a = b$$

Where K and k are constants.

7. Let $f: R \to (-1,1)$ be defined by

$$f(x) = \frac{x}{1+|x|}, \quad x \in R$$

Show that f is continuous, one-one, onto and has a continuous inverse.

8. Let $f_n: R \to R$ be defined by

$$f_n(x) = \frac{2x^2}{3x^2 + (4-5nx)^2}$$

Check whether

(a) $\{f_n\}$ is point wise convergent and

(b) $\{f_n\}$ is uniformly convergent.

9. Find the general solution of the differential equation $(1-x^2) y'' - 3xy' - y = 1$.

10. Solve for x the system $x' = Ax$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

11. Let R be the group of all real numbers under addition and Z be the subgroup of all integers. Show that R/Z is isomorphic to the group of all complex numbers with absolute value 1, under multiplication.

12. Let R^3 be the ring with addition and multiplication defined by

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$xy = (x_1 y_1, x_2 y_2, x_3 y_3)$$

For $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Let $I = \{x \in R^3 : x_3 = 0\}$.

Show that I is a maximal ideal of R^3.

13. Let $\{u_1, u_2, ...\}$ be an orthonormal set in a Hilbert space H. Prove that for every $x \in H$, the series

$$\sum_{j=1}^{\infty} \langle x, u_j \rangle u_j$$

Is convergent in H and

$$\left\| x - \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j \right\| \le \| x - u \|$$

For every $u \in$ span $\{u_1, u_2, ...\}$

14. Find a, b and c so that the numerical integration formula

$$\int_{-h}^{h} f(t)\, dt = af(-h) + bf(0) + cf(h)$$

Is exact for all polynomials of degree ≤ 2. Give an example of a polynomial of the lowest degree for which the formula is not exact.