MATHEMATICS

Duration: Three Hours *Maximum Marks: 150*

- **Read the following instructions carefully**
  1. Write all the answers in the answer book only.
  2. This question paper consists of TWO SELECTIONS: A and B.
  3. Section A has SEVEN questions. Answer ALL questions in this section. Answers to questions in this section must be written only on the first ten pages of the answer book. The first two pages of the answer book show boxes where the student should give answers to multiple choice questions.
  4. Section B has Twenty questions. Answer any TEN questions from this section. If more questions are attempted, score off the answers not to be evaluated; else only First Ten un scored answers will be considered.
  5. Answers to Section B should start on a fresh page, and should not be mixed with answers to Section A.
  6. In all question carrying five marks each, write clearly the important steps in your answers. These steps carry credit.
  7. Answers to questions and answers to parts of a question should appear together and should not be separated.
  8. There will be no negative marking.
- **Note**

  The symbols $\Re, \Re^n, \Re^{m\times n}, C^{m\times n}$ respectively denote the real line, the set of n-component column vector over $\Re$ the set of m × n matrices over δ and the set of m × n matrices over C, the set of complex number. $C^1[a,b]$ is the set of continuously differentiable functions on $[a,b]$ and $C^2[a,b]$ the set of twice continuously differentiable functions on $[a,b]$.

SECTION – A (100 Marks)

1.1 If $f(x)$ is real valued function defined on $[0,\infty]$ such that $f(0)=0$ and $f''(x)>0$ for all x, then the function $h(x)=\dfrac{f(x)}{x}$ is

(A) Increasing in $[0,\infty]$

(B) Decreasing in $[0,1]$

(C) Increasing in $[0,1]$ and decreasing in $[1,\infty]$

(D) Decreasing in $[0,1]$ and increasing in $[1,\infty]$

1.2 $\dfrac{d}{dx}\displaystyle\int_{\sin^2 x}^{2\sin x} e^{t^2}\,dt$ at $x=\pi$ is

(A) 1

(B) –1

(C) 2

(D) –2

1.3 Let $f(x,y)=\sqrt{|xy|}$. Then

(A) f_x and f_y do not exist at $(0,0)$

(B) $f_x(0,0)=1$

(C) $f_y(0,0)=0$

(D) f is differentiable at $(0,0)$

1.4 In a metric space (x,d)

(A) Every infinite set E has a limit point in E

(B) Every closed subset of a compact set is compact

(C) Every closed and bounded set is compact.

(D) Every subset of a compact set is closed.

1.5 Let (x,d) be a complete metric space and $f:X\to X$ satisfies $df(x), f(y)\le\alpha(x,y)$ for some $\alpha, 0\le\alpha<1$ for all $x,y,\in X$.

(A) f is bounded function on X

(B) f need not be continuous on X.

(C) $\{f(X_n)\}_{n=1}^{\infty}$

(D) $f(p)=p$ for some $p\in X$.

1.6 Let $f_n(x)=\begin{cases}1/x & \text{If } n<x<n+1\\ 0 & \text{otherwise}\end{cases}$

If $f(x)\lim_{x\to\infty} f_n(x)$, then $\int_{(0,\infty)} f(x)dx = \lim_{x\to\infty}\int_{(0,\infty)} f_n(x)dx$ follows by

(A) Bounded Convergence Theorem
(B) Monotone Convergence Theorem
(C) Dominated Convergence Theorem
(D) None of the above

1.7 Let $f_x(x)=\frac{\sin x}{\sqrt{n}}, n=1,2,....$ and $x\in[-1,1]$. Then as $n\to\infty$,

(A) $\{f_n(x)\}_{n=1}^{\infty}$ does not converge uniformly in $[-1,1]$

(B) $\lim_{n\to\infty}\int_{-1}^{1} f_n(n)dx \neq 0$

(C) $\{f_n^{'}(x)\}$ does not converge uniformly in $[-1,1]$

(D) $f_n(x), n=1,2$ is not uniformly continuous in $[-1,1]$

1.8 The function $w(z)=-\left(\frac{1}{z}+bz\right), -1<b<1,$ maps $|z|<1$ on to

(A) A half plane
(B) Exterior of the circle
(C) Exterior of an Ellipse
(D) Interior of an ellipse

1.9 The function

$$f(z)=\begin{cases}(\bar{z})^2/z^2 & \text{If } z\neq 0\\ 0 & \text{If } z=0\end{cases}$$

(A) Satisfies the Cauchy-Reimann equations at z = 0
(B) Is not continuous at z = 0
(C) Is differentiable at z = 0
(D) Is analytic at z =1

1.10 The value of $\int_{|z|=2}\frac{e^2}{(z+1)^2}dz$ is

(A) $2\pi ie^{-1}$

(B) $\frac{8\pi i}{3}e^{-2}$

(C) $\frac{2\pi i}{3}e^{-2}$

(D) 0

1.11 Let (x,τ) be a topological space, where $X=\{a,b,c,d\}$ and $\tau=\phi X,\{a\},\{a,b\},\{a,c\}\{a,b,c\}$. The limit points of the Set A = $\{a,c,d\}$ are

(A) a and b
(B) b and c
(C) c and d
(D) d and a

1.12 Let A and B be $n\times n$ matrices with the same minimal polynomial. Then

(A) A is similar to B
(B) A is diagonalizable if B is diagnosable
(C) A–B is singular
(D) A and B commute

1.13 Let $A\in C^{m\times n}$ and A' A* denote respectively the transpose and conjugate transpose of A.

Then

(A) ran (AA*A) = rank (A)
(B) rank (A) = rank (A2)
(C) rank (A) = rank (A'A)
(D) rank (A^2) – rank (A) = rank (A^3) – rank (A^2)

1.14 Consider 2 × 2 matrix, $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

If $a+d=1=ad-bc,$ then A^3 equals

(A) 0
(B) –1
(C) 31
(D) None of these

1.15 Let $C_{2\pi}$ denote the set of all 2π periodic continuous functions. Let S_f denote the Fourier series of f. That $S_f\notin C_{2\pi}$ for some $f\notin C_{2\pi}$, follows from

(A) Principle of Uniform Bound ness
(B) Hahn Banach theorem
(C) Open mapping theorem
(D) Closed graph theorem

1.16 The sequence $\{x_n\}$ of $m\times m$ matrices defined by the iterations

$X_{n+1}=2X_n-X_{n'}\quad AX_{n'}\quad n=0,1,2.$

When $X_0 = I$, the identity matrix converges to A^{-1}, if each eigen value λ of A satisfies

(A) $|\lambda| < 1$

(B) $|\lambda - 1| < 1$

(C) $|\lambda + 1| < 1$

(D) None of the above

1.17 A nontrivial solutions of

$x^2 y'' + xy' + 4y = 0, \quad x > 0$ are

(A) Bounded and non periodic

(B) Un bounded and non-periodic

(C) Bounded and periodic

(D) Unbounded and periodic

1.18 Every solution of $y'' + ay' + by = 0$, where a and b are constants, approximate to zero as $x \to \infty$ provided

(A) $a > 0, b > 0$

(B) $a > 0, b < 0$

(C) $a < 0, b < 0$

(D) $a < 0, b > 0$

1.19 The boundary value problem

$y'' + ky = 0, \; y'(0) = 0, \; y'(\pi) = 0$

Has non trivial solution for

(A) All negative value of k

(B) All the values of k

(C) k = 0

(D) $k = \pm n, n = 1, 2, \ldots,$

1.20 Let $f(x,y) = x - y$ and c be the boundary of the square $S = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.

Then $\int_c f \dfrac{\partial f}{\partial n} ds$ equals

(A) 0

(B) 1

(C) 2

(D) None of these

1.21 The work done in moving a particle in the force field $\bar{F} = 5x^2\hat{i} + (xz - y)\hat{j} + 3z\hat{k}$ along the straight line joining (0,0,0) to (1,1,1) is

(A) 0

(B) 1

(C) 2

(D) 4

1.22 The linear programming problem Max. $(-3x_1 + x_2)$ Subject to the constraints $2x_1 + x_2 \le 4$, $-x_1 + x_2 \le 1$, $x_1 + x_2 \ge 1$, $-3x_1 + x_2 \ge -3$, where $x_1, x_2 \ge 0$ has

(A) Unique of optimal solution

(B) Alternative optimal solution

(C) Constraints leading to an infeasible region

(D) Constraints leading to an unbounded region

1.23 If X and Y are independent N(0,1) variables, then the characteristic function of XY is,

(A) e^{-t^2}

(B) $(1 + t^2)^{-1/2}$

(C) $(1 + t)^{1/2}$

(D) $(1 + t^2)^{-1}$

1.24 Let $F(x,y) = \begin{cases} 0 & \text{If} \quad y < e^{-x} \\ 1 & \text{If} \quad y \ge e^{-x} \end{cases}$

Then F is not a probability distribution function because

(A) It is discontinues

(B) It is not Monotone in x and in y

(C) It is not right continuous in x and y

(D) It does not satisfy

1.25 $\lim\limits_{n \to \infty} e^{-n} \sum\limits_{k=n}^{\infty} \dfrac{n^k}{k!}$

(A) Is 0

(B) Is 1

(C) Is $\dfrac{1}{2}$

(D) Does not exist

2.1 The largest internal in which $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{x^n}{n}$ converges is

(A) (–1,1)

(B) [–1,1)

(C) (–1,1)

(D) [–1,1]

2.2 $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \dfrac{1}{3n + k}$ is

(A) log 4/3

(B) log 3/4
(C) log 3/2
(D) log 5/4

2.3 Let $E_k, k = 1, 2, ...n$, be pair wise disjoint subset of real line R with Lebesgue measure $m(E_k) = k^3$. If $E = \bigcup_{k=1}^{n} E_k$ and $\phi(x) = \sum_{k=1}^{n} \frac{1}{k} X_{E_k}$, where $X_{(E_k)}$ is the characteristic event function of E_k, then the Lebesgue integral $\int_E \phi \, dx$ equals

(A) 0
(B) Log n
(C) $\frac{n(n+1)}{2}$
(D) $\frac{n(n+1)(2n+1)}{6}$

2.4 The conjugate (also called symmetric) point of 1 + i with respect to the circle $|z-1| = 2$ is

(A) $1-i$
(B) $1+4i$
(C) $1+2i$
(D) $-1-i$

2.5 The residue of $f(z) = \cot z$ at any of its poles is

(A) 0
(B) 1
(C) $\sqrt{3}$
(D) None of these

2.6 The harmonic conjugate of $u(x, y) = x^2 - y^2 + xy$ is

(A) $x^2 - y^2 - xy$
(B) $x^2 + y^2 - xy$
(C) $2xy + \frac{1}{2}(y^2 - x^2)$
(D) $\frac{1}{2}xy + 2(y^2 - x^2)$

2.7 The singularity of $e^{\sin z}$ at $z = \infty$ is

(A) A pole
(B) A removable singularity
(C) Non isolated essential singularity
(D) Isolated essential singularity

2.8 Let $d < x, y >= |x_1 - y_1| + |x_2 - y_2|$ be a metric defined on $\Re^2$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then the set $S = [x \in \Re^2 : d < x, 0 >= 1)$ is

(A) A circle
(B) An ellipse
(C) A rhombus
(D) A rectangle

2.9 Let $A \in \Re^{n \times n} B \in \Re^n$. Consider the two systems of linear equations,

(I) $Ax = b$ and
(II) $A'Az = A'b$

Then

(A) Both (I) and (II) have the same solution
(B) (I) has more solutions than (II)
(C) Both (I) and (II) may not have any solution
(D) (I) may not have any solution but (II) always has at lest one solution.

2.10 Let A be an n × n matrix which is both Hermitian and unitary. Then

(A) $A^2 = I$
(B) A is real
(C) The eigen value of A are 0,1,–1
(D) The characteristic and minimal polynomials of A are the same.

2.11 There exist zero-divisors in

(A) The ring of integers modulo a prime p
(B) The ring of real matrices of order p
(C) The ring of polynomials and a field of characteristics
(D) The ring of entire functions

2.12 Let G be a group such that $a^2 = e$ for each $a \in G$, where e is the identity element of G. Then

(A) G is cyclic
(B) G is finite
(C) G is abelian
(D) G has a subgroup which is not normal

2.13 The Lagrange interpolation of the function $f(x) = x^2$ on the interval $[-1, 1]$ has the least maximum error, if the nodes of interpolation are

(A) Zeros of the leg ender polynomials, P_n

(B) Zeros of Chebyshev polynomials, I_n

(C) $\{k/(n-1): 0 \le k \le n-1\}$

(D) $\{k/(n+1): 1 \le k \le n\}$

2.14 The following matrix admits a Cholesky (also called LL*) decomposition:

(A) $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

2.15 The value of the line integral $\oint_c \frac{-y\,dx + x\,dy}{x^2+y^2}$, where c is the unit circle centre at 0, equals,

(A) 2π

(B) -2π

(C) 0

(D) None of these

2.16 If $(f(x,y,z)$ is a harmonic function in a domain D containing the region $T + x^2 + yh2 + z^2 \le 100$, then the surface integral $\int_s \left(x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \right) ds,$ where $s: x^2 + y^2 + z^2 = 100$, equals

(A) 0

(B) 1

(C) 10

(D) 100

2.17 Let R be the image of the triangular region S with vertices (0,0) (1,0) and (0,1) is uv-plane under the transformation $x = 2u - 3v,\ \ y = u + v,$

Then $\iint_R x\,dA$ equals

(A) $\iint_s (2u-3v)\,ds$

(B) $3\iint_s (2u-3v)(u+v)\,ds$

(C) $-\iint_s (2u+3v)\,ds$

(D) $5\iint_s (2u-3v)\,ds$

2.18 If $y_1(x)$ and $y_2(x)$ are solutions of $y'' + xy' + (1-x^2)y = y\sin x$

Then which of the following is also its solution?

(A) $y_1(x) + y_2(x)$

(B) $y_1(x) - y_2(x)$

(C) $2y_1(x) - y_2(x)$

(D) $y_1(x) - 2y_2(x)$

2.19 If $y_1(x)$ and $y_2(x)$ are solutions of $y'' + x^2y' + (1-x)y = 0$ such that $y_1(0) = 0,\quad y'=(0) = -1$ and $y_2'(0) = y_2(0) = 1$ then the Wronskian $W(y_1, y_2)$ on R

(A) Is never zero

(B) Is identically zero

(C) Is zero only at a finite number of points

(D) Is zero at count ably infinite number of points

2.20 For the differential equation $xy' - y = 0$

Which of the following function is not an integrating factor?

(A) $\frac{1}{x^2}$

(B) $\frac{1}{y^2}$

(C) $\frac{1}{xy}$

(D) $\frac{1}{x+y}$

2.21 The partial differential equation $y^3u_{xx} - (x^2-1)u_{yy} = 0,$ is

(A) Parabolic in $\{(x,y): x<0\}$

(B) Hyperbolic in $\{(x,y): y>0\}$

(C) Elliptic in $\Re^2$

(D) Parabolic in $\{(x,y): x>0\}$

2.22 The partial differential equation of the family of surfaces $z=(x+y)+A(xy)$ is

(A) $xp-yq=0$

(B) $xp-yq=x-y$

(C) $xp+yq=x+y$

(D) $xp+yq=0$

2.23 A particle moves in the xy–plane under the influence of a central force which varies inversely to its distance from the centre of force. If T is the kinetic energy, V is the potential energy, L is the Lagrangian and H is Hamiltonian of the system, then

(A) $L=H-V$

(B) $H=T-V$

(C) $H=T+V$

(D) None of these

2.24 Which of the following statements is the strongest?

(A) X_n converges to X in probability

(B) X_n converges to X in distribution

(C) A subsequence of $\{X_n\}$ converges to X with probability 1

(D) $E\left(X_n-X^2\right)\to 0$ as $n\to\infty$.

2.25 $P(A|B)+P(A|B^c)$ is

(A) P(A)

(B) 1

(C) Greater than P(A)

(D) None of these

1. Let $f:(0,\infty)\to\Re$ be a twice differentiable function and $|f(x)\le 1,|f''(x)|\le 2$ for all x. Show that $|f'(x)|\le 2\sqrt{2}$ for all x.

2. Let $f(x)$ be an entire function satisfying $|f(z)|\le k|z|^2$ for some constant k and all z. Show that $f(z)=az^2$ for some constant a.

3. Find the eigen values of the n × n matrix.

$$A=\begin{bmatrix} 1 & 1 & \dots\dots\dots & 1 \\ 1 & 1 & \dots\dots\dots & 1 \\ \dots & \dots & \dots\dots\dots & \dots \\ 1 & 1 & \dots\dots\dots & 1 \end{bmatrix}_{n\times n}$$

4. Let y(t) be the solution of the initial value problem:

$$t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty=0,\ \ y(0)=2,\ \ y'(0)=0$$

Find the Lap lace transform of y(t)

5. Let $(\Omega, Au_1), i=1,2$ be probability spaces and $u_1(A)\int_A f_i d\mu$ for $A\in A$. Prove that

Sub $\underset{A\in}{|}\ u_1(A)-u_2(A)|=\frac{1}{2}\int_{\Omega}|f_1-f_2|\,d\mu$

SECTION – B (50 Marks)

Answer any TEN questions. Each question carries 5 marks.

6. Test the function $f(x,y)=3x^4-7x^2y+2y^2$ for minimum at (0,0)

7.

8. Using the contour integration evaluate $\int_0^\infty \frac{\cos 2x}{x^2+16}dx$

9. Show that if a topological space X is Hausdorff, then the diagonal $\Delta=\{(x,x):x\in X\}$ is X^2 is a closed set.

10. Prove every orthonormal set in $L^2[0,1]$ is countable

11. Find the general solution of the differential equation $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$.

12. Using $\eta=x+y$ as one of the trans formation variables obtain the --@-- from of

$$\frac{\partial^2 4}{\partial x^2}=\frac{2\partial^2 4}{\partial x\partial}+\frac{\partial^2}{\partial}$$

13. Let (x,t) be the solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial^2}$$

$$u(x,0)=u_0(x)$$

$$\left(\frac{\partial u}{\partial}\right)_{t=0} = u_1(x)$$

And $u(a,t) = u(b,t) = 0$ for $t \geq 0$.

If $u_0(x) \in = C^2[a,b]$ show that

$E(t) = \int_a^b \left[\left(\frac{\partial \phi}{\partial}\right)^2 + \left(\frac{\partial \phi}{\partial}\right)^2\right] dx$ is constant

$u(b,t) = 0$ for $t \geq 0$;

14. Let a > 0. Prove that Newton's method: $x_{n+1} = x_n(2 - ax_n), n = 0,1,2,....$ for calculating $\frac{1}{a}$ as a zero of $f(x) = \frac{1}{x} - a$ converges to $\frac{1}{a}$ if and only if $x_0 \in \left(0, \frac{2}{a}\right)$.

15. Find such that the quadrature formula

$$\int_0^1 \frac{f(x)}{\sqrt{x}} dx \cong Af(0) + Bf(\lambda) + Cf(1)$$

may be exact for polynomials of degree 3.

16. Show that any maximal ideal in the cumulative ring $F[x]$ of polynomial over a field F is the principle ideal generated by an irreducible polynomial.

17. If $A \in \Re^{m \times m}$ and B $\Re^{n \times m}$ have a common eigen value $\lambda \in \Re$, show that the linear operator $\Re^{m \times n} \to \Re^{m \times n}$, defined by TX = AX – XB, is singular.

18. Consider the LPP's

Minimize $(-1 + 2\lambda x_1 + (-3 + \lambda)x_{2'}$

Such that $x_1 + x_2 \leq 6$

$-x_1 + 2x_2 \geq 0$

Where $\lambda \geq 0$ is a parameter.

Using simplex method show that $(x_1, x_2) = (2,4)$ provides the optimal solution when $\lambda = 0$. Determine all the other values of λ for which (2,4) continues to provide the optimal solution.

19. A particle P slides on a smooth wire bent in the form of a vertical circle of radius a. The wire rotates about a fixed vertical diameter AB with uniform angular velocity ω. If O is the centre of the circle and at time $t, \angle AOP = \theta(t)$, set up the Lagrangian of the system and show that

$\theta^2 + \omega^2 \cosh 2\theta - \frac{2g}{a}\cos\theta$ is a constant.

20. Consider force free rotational motion of an axially symmetric rigid body, that has a fixed point on the axis of symmetry. Using Euler's equation of motion, show that the angular velocity has constant magnitude.

21. Prove that $e^{\cos t - 1}$ is a characteristic function.

22. Let X_1 and X_2 be independent Bernoulli random variables with parameter $\theta, 0 < \theta < 1$. 1s $T(x_1, x_2) = x_1 + x_2 + \text{Max}(x_1, x_2)$? Why?

23. Let X have a binomial distribution with

$$P(X = x) = \binom{10}{x}\theta^x(1-\theta)^{10-x},$$

$$x = 0,1,.....,10; \frac{3}{4} \leq \theta \leq 1$$

Obtain the maximum likelihood estimate of θ, if x = 4.

24. Let $(x_1........, x_n)$ be a random sample from a $N(\mu, \sigma^2)$ population. Using completeness and sufficiency of $\left(\bar{x}, \sum x_i^2\right)$ obtain

$$E\left(\frac{(x_1 - x_2)^2}{2} \mid \bar{x}, \sum x_i^2\right)$$

25. Let x be a random variable with the probability function f_0 under the null hypothesis H_0 and f_1 under the alternative hypothesis H_1. The functions f_0 and f_1 are given by

$f_0(0) = .2$

$f_0(1) = .3$

$f_0(2) = .3$

$f_0(3) = .2$,

$f_1(0) = .3$

$f_1(1) = .45$

$f_1(2) = .15$

$f_1(3) = .1$

Obtain the class of the most powerful tests of level .01. What is the power of any member of this class?