MATHEMATICS

Duration: Three hours *Maximum Marks: 150*

- **Note:** The symbols R, Q, I and C denote the set of all real, rational, integer and complex numbers respectively. Vector quantities are denote by bold letters.

SECTION A (75 Marks)

1. This question consists of Thirty (30) sub-questions, each carrying two marks. For each sub-question one or more of the suggested alternatives may be correct. The alphabet corresponding to the correct alternative(s) MUST be written only in the boxes corresponding to the questions in the first sheet of the answer book.

1.1 Let W be the space spanned by $f=\sin x$ and $g=\cos x$. Then for any real value of θ, $f_1=\sin(x+\theta)$ and $g_1=\cos(x+\theta)$.

(a) Are vectors in W

(b) Are linearly independent

(c) Do not form a basis for W

(d) Form a basis for W

1.2 Consider the basis $S=\{v_1,v_2,v_3\}$ for R^3 where $v_1=(1,1,1),V_2=(1,1,0)$, $v_3=(1,0,0)$ and let $T:R^3\rightarrow R^2$ be a linear transformation such that $Tv_1=(1,0),Tv_2=(2,-1),Tv_3=(4,3)$. Then $T(2,-3,5)$ is

(a) (– 1,5)

(b) (3,4)

(c) (0,0)

(d) (9,23)

1.3 For $0<\theta<\pi$, the matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

(a) Has no real eigen value

(b) Is orthogonal

(c) Is symmetric

(d) Is skew symmetric

1.4 For the function $f(z)=\dfrac{1-e^{-z}}{z}$, the point z = 0:

(a) An essential singularity

(b) A pole of order zero

(c) A pole of order one

(d) Is skew symmetric

1.5 The transformation $w=e^{i\theta}\left(\dfrac{z-p}{\overline{p}z-1}\right)$ where p is a constant, mops $|z|<1$ on to

(a) $|w|<1$ if $|p|<1$

(b) $|w|>1$ if $|p|>1$

(c) $|w|=1$ if $|p|=1$

(d) $|w|=3$ if p = 0

1.6 The value of the integral $\oint_c \dfrac{dz}{z^2-1},C:|z|=4$ $\oint_c \dfrac{dz}{z^2-1},C:|z|=4$ is equal to

(b) 0

(d) $2\pi i$

1.7 Let E be the set of all rational p such that $2<p^2<3$. Then E is

(a) Compact in Q

(b) Closed and bounded in Q

(c) Not compact in Q

(d) Closed and unbounded in Q

1.8 Let A be the set of points in the interval (0,1) representing the numbers whose expansion as infinite decimals do not contain the digit 7. Then the measure of A is

(a) 1

(b) 0

(c) $\frac{1}{2}$

(d) ∞

1.9 The Fourier expansion in the interval $[-4,4]$ of the function

$f(x) = -, -4 \le x \le 0,$

$= x, 0 \le x \le 4,$ has

(a) No cosine term

(b) No sine term

(c) Both cosine and sine terms

(d) None of these

1.10 The particular solution of the equation $y' \sin x = y$ in y satisfying the initial condition $y\left(\frac{\pi}{2}\right) = e,$ is

(a) $e^{\tan(x/2)}$

(b) $e^{\cot(x/2)}$

(c) In $\tan\left(\frac{x}{2}\right)$

(d) In $\cot\left(\frac{x}{2}\right)$

1.11 The differential equation $\frac{dy}{dx} = h(a-y)(b-y),$ when solved with the condition $y(0) = 0$, yields the result

(a) $\frac{b(a-y)}{a(b-y)} = e^{(a-b)kx}$

(b) $\frac{b(a-x)}{a(b-x)} = e^{(b-a)ky}$

(c) $\frac{a(b-y)}{b(a-y)} = e^{(a-b)kx}$

(d) $xy = ke$

1.12 The Sturm-Liouville problem: $x'' + \lambda^2 y = 0,\ y'(0) = 0, y'(\pi) = 0$ has its eigenvectors given by y =

(a) $\sin\left(n + \frac{1}{2}\right)x$

(b) $\sin nx$

(c) $\cos\left(n + \frac{1}{2}\right)x$

(d) $\cos nx;$ where $n = 0, 1, 2, \ldots$

1.13 Let G be the additive group of integers I and G' be the multiplicative group of the fourth roots of unity. Let $f : G \to G'$ be a homomorphism mapping given by f(n) = i^n; where $i = \sqrt{-1}$. Then the Kernel of f is

(a) empty set

(b) $\{4m; m \in I\}$

(c) $\{(2m)^2 + 1 : m \in I\}$

(d) $\{2m + 1 : m \in I\}$

1.14 If A is the subspace of l^∞ consisting of all sequences of zeros and ones and d is the induced metric on A, then the rare sets in (A,d) are

(a) Empty set

(b) All singleton subsets of A

(c) Power set of A

(d) Set A itself

1.15 The norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)\,dt$$ is

(a) Zero

(b) One

(c) Two

(d) Three

Where $C[-1,1]$ denotes a Banach space of all real valued functions x(t) on $[-1,1]$ with norm given by $\|x\| = \max_{t\in[-1,1]} |x(t)|$.

1.16 The smallest value of $x\left(|x|<1\right)$ correct to two decimal places satisfying the equation

$$x-\frac{x^3}{3}-\frac{x^5}{10}-\frac{x^7}{42}+\frac{x^9}{216}-\frac{x^{11}}{1320}+....$$
$$=0.4431135$$

Is

(a) 0.58

(b) 0.47

(c) 0.44

(d) 0.88

1.17 The Jacobi's iteration method for the set of equations

$$x_1+ax_2=2, 2ax_1+x_2=7, \left(a\neq\frac{1}{\sqrt{2}}\right)$$

converges for

(a) All values of a

(b) a = 1

(c) $|a|<\frac{1}{\sqrt{2}}$

(d) $\frac{1}{\sqrt{2}}<a<\sqrt{\frac{3}{2}}$

1.18 The interpolating polynomial of highest degree which corresponds the functional values
$f(-1)=9, f(0)=5, f(2)=3, f(5)=15,$
is

(a) x^3+x^2+2x+5

(b) x^2-3x+5

(c) $x^4+4x^3+5x^2+5$

(d) $x+5$

1.19 The equation

$$x^2(y-1)Z_{xx}-x(y^2-1)Z_{xy}+y(y^2-1)Z_{yy}+Z_x=0$$

is hyperbolic in the entire xy–plane except along

(a) x–axis

(b) y–axis

(c) A line parallel to y–axis

(d) A line parallel to x–axis

1.20 The solution of the Cauchy problem

$$u_{yy}(x,y)-u_{xx}(x,y)=0;$$

$u(x,0)=0, u_y(x,0)=x$ is $u(x,y)=$

(a) $\frac{x}{y}$

(b) xy

(c) $xy+\frac{x}{y}$

(d) 0

1.21 The characteristics curves of the equation

$$x^2u_{xx}-y^2u_{yy}=x^2y^2+x;\ \ x>0, u=(x,y)$$

are

(a) Rectangular hyperbola

(b) Parabola

(c) Circle

(d) Straight line

1.22 The number of generalized coordinates for a pair of scissors that can move in a plane is

(a) 1

(b) 2

(c) 3

(d) 4

1.23 The topology is τ on the real line R generated by the class $\Im$ of all closed intervals $[d, d+l]$ with length l is

(a) Indiscrete

(b) Discrete

(c) Standard topology

(d) Neither discrete nor Housdorff

1.24 Let norms $\|x\|_1=\sum_{i=1}^{n}|\xi_i|$ and $\|x\|_2=\left(\sum_{i=1}^{n}|\xi_i|^2\right)^{\frac{1}{2}}$ induce topologies τ_1 and τ_2 on R^n the n-dimensional Euclidean space, then

(a) τ_1 is weaker than τ_2

(b) τ_1 is stronger than τ_2

(c) τ_1 is equivalent to τ_2

(d) τ_1 and τ_2 are incomparable

1.25 he probability that exactly one of the events E or F occurs is equal to

(a) $P(E)+P(F)-P(EF)$

(b) $P(E)+P(F)-2P(EF)$

(c) $P(EF^c)+P(E^cF)$

(d) $P(E)+P(F)$

1.26 If $\{A_n, n\geq 1\}$ is a sequence of events, then $\lim\limits_{n\to\infty} P(A_n) = P\left(\lim\limits_{n\to\infty} A_n\right)$ if

(a) $\{A_n\}$ is an increasing sequence of events

(b) $\{A_n\}$ is a decreasing sequence of events

(c) $\{A_n\}$ is neither increasing nor decreasing sequence of events

(d) None of these

1.27 Suppose that the five random variables $X_1,......,X_5$ are independent and each has standard normal distribution. A constant C such that the random variable $\dfrac{C(X_1+X_2)}{(X_3^2+X_4^2+X_5^2)^{\frac{1}{2}}}$ will have a t-distribution, has the value

(a) $\dfrac{\sqrt{3}}{2}$

(b) $\sqrt{\dfrac{3}{2}}$

(c) $\dfrac{3}{2}$

(d) $\sqrt{\dfrac{2}{3}}$

1.28 Suppose that $X_{1},......,X_n$ are random variables such that the variance of each variable is 1 and the correlation between each pair of different variables is $\dfrac{1}{4}$ Then Var $(X_1+X_2+......+X_n)$ is

(a) $\dfrac{n(n+1)}{2}$

(b) $\dfrac{n(n+2)}{4}$

(c) $\dfrac{n(n+3)}{4}$

(d) $\dfrac{n(n+3)}{2}$

1.29 The objective function of the dual problem for the following primal linear programming problem:

Maximize $f=2x_1+x_2$

Subject to $x_1-2x_2\leq 2,$

$x_1+2x_2=8,$

$x_1-x_2\leq 11,$

With $x_1\geq 0$ and x_2 unrestricted in sign, is given by

(a) Minimize $z=2y_1-8y_2+11y_3$

(b) Minimize $z=2y_1+8y_2+11y_3$

(c) Minimize $z=2y_1-8y_2-11y_3$

(d) Minimize $z=2y_1+8y_2-11y_3$

1.30 If $y(t)=1+\int_0^t y(v)e^{-(t+v)}dv$ then $y(t)$ at t = 1 equals

(a) 0

(b) 1

(c) 2

(d) 3

2. This question consists of Five sub-questions, carrying Three marks each.

All the sub-questions are to be answered.

(5×3)

2.1 Expand the function $f(z)=\frac{1}{3-2z}$ in powers of $(z-3)$ and find the radius of convergence of the series so obtained.

2.2 Develop the Hamiltonian and hence obtained the canonical equations of motion for a system for which Lagrangian $L=\frac{1}{2}mq^2-\frac{\mu}{2}q^2$, where $\dot{q}=\frac{dq}{dt}$, q being the generalized co-ordinate of the system.

2.3 Evaluate $\int_0^{\frac{\pi}{2}}\sqrt{1-0.162\sin^2\phi d\phi}$ by Simpson's one-third rule by taking the step sizes as $\frac{\pi}{12}$.

2.4 Show that in an inner product space, $x\perp y$ (x is orthogonal to y) if and only if we have $\|x-\alpha y\|=\|x-\alpha y\|$ for all scalars α.

2.5 Let X and Y have joint probability density function

$f_{X,Y}(x,y)=2e^{-(x+y)},0<x<y,<y$. Find $P(Y<3X)$.

SECTION B (75 Marks)

This section consists of Twenty questions of five marks each. Any Fifteen out of them have to be answered. If more number of questions are attempted, score off the answers not to be evaluated, else only the first fifteen un scored answers will be considered strictly. (15 × 5 = 75)

3. Solve the following linear programming problem using the Simplex method:

A. Minimize $f=-40x_1-100x_2$

Subject to

$10x_1+5x_1\leq 2500,$

$4x_1+10x_2k,2000,$

$2x_1+3x_2\leq 900,$

$x_1\geq 0, x\geq 0$

(5)

4. Evaluate the integral $\int_c \frac{e^{1/z^2}}{z^2+1}dz; C:|z-i|=\frac{7}{2}$, where integration is to be taken counterclockwise. (5)

5. (a) Construct an analytic function $f(z)$ of which the real part is $u(x,y)=2xy+\cos hx\sin y,$ given that $f(0)=0$. (2)

(b) Determine all harmonic functions of the form $u=f\left(\frac{x^2+y^2}{x}\right)$ that are not constant.

6. Evaluate $\iint_s (curl\, v).ndS$

Where $v=2yi+3xj-z^2k$ and S is the upper half surface of the sphere $x^2+y^2+z^2=9$, n is a positive unit normal vector to S and C is its boundary.

(5)

7. (a) Examine the series $\sum_{n=1}^{\infty}\frac{nx}{n+x}, x\in[0,1]$ as regards to its uniform convergence on the domain $0\leq x\leq 1$. (3)

(b) Prove that set of points on which a sequence of measurable functions (f_n) converges is measurable.

8. Use the Laplace transform procedure to solve the initial value problem:

$ty''(t)+y'(t)+ty(t)=0; y(0)=1, y'(0)=k,$

(k is a constant) (5)

9. Construct the Green's function for the boundary value problem:

$y''(x)+y(x)=-1; y(0)=0, y\left(\frac{\pi}{2}\right)=0$

and hence solve the equation. (5)

10. Prove that the set A of all 2 × 2 matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ where a, $b\in I$, the set of integers, is a left ideal but not a right ideal in the ring $\Re$ of all 2 × 2 matrices over I.

(5)

11. Show that if p is a prime number, then any group G of order 2p has a normal subgroup of order p. (5)

12. If (x_n) in a Banach space $(x\|.\|)$ is such that $(f(x_n))$ is bounded for all $f \in X'$ the dual of X, then using uniform bounded ness theorem, show that $(\|x_n\|)$ is bounded. (5)

13. Let X be the nor med space whose points are sequences of complex numbers $x = (\xi_j)$ with only finitely many non-zero terms and defined by $\|x\| = j\sup|\xi_j|$ Let $T : X \to X$ is defined by

$$y = Tx = \left(\xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, ...\right),$$

Show that T is linear and bounded but T^{-1} is unbounded. Does this contradict the open mapping theorem? (5)

14. Determine the step-size that can be used to evaluate the integration $\int_1^2 \frac{dx}{x}$ by using Simpson's one-third rule so that the truncation error is less than 5×10^{-4} and hence evaluate the integral. (5)

15. Solve

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y; y(0) = 3, y'(0) = 0,$$

To approximate $y(0,1)$ by using fourth order Runge-Kutta method. (5)

16. Use Lagrange's method to solve the equation.

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix} = 0; \text{ where } z = z(x, y) \quad (5)$$

17. (a) Reduce the equation $u_{xx}(x, y) - u_{yy}(x, y) - \frac{2}{x}u_x = 0$ to a possible canonical form. (3)

(b) Reduce the following heat conduction problem with non-homogeneous boundary conditions:

$$u_t(x,t) - u_{xx}(x,t) = 0; 0 < x < \pi, t > 0,$$

$$u(0,t) = 0,\ u(\pi,t) = 10,$$

$$u(x,0) = f(x)$$

To a problem with homogeneous boundary conditions. (2)

18. (a) A smooth circular wire rotates with constant angular velocity V about the vertical axis CA which lies on the plane of the circle and passes through the point C on the diameter of the circle. A particle P slides on the wire. Taking PC = R and $\angle PCA = \theta$, develop the Lagrangian of the system and hence the equations of motion if the generalized force-components are Q_R and Q_θ. (3)

(b) A system moves in a force-free space with kinetic energy by $T = q_i\dot{q}_i - \sqrt{1 - q_i^2}$, where q_i are the generalized coordinate of the system, q_i the generalized velocity components. Show that the generalized accelerations $\ddot{q}_i$ vanish.

19. Let A be a subset of a topological space (X, τ) and τ_A be the relative topology on A. Then A is τ–connected if and only if A is $\tau_A -$ connected. Prove this. (5)

20. Let $X_1, X_2, ..., X_n$ be a random sample from a poisson distribution with parameter $\theta > 0$. Find the uniformly minimum variance unbiased estimator (UMVUE) of P $(X \le 1) = (1+\theta)e^{-\theta}$.

21. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution that is normally distributed with mean θ_1 and variance θ_2. Find a best test of the simple hypothesis $H_0 : \theta_1 = 0,\ \theta_2 = 1$ against the alternative simple hypothesis $H_1 : \theta_1 = 1, \theta_2 = 4.$ (5)

22. (a) Solve the integral equation $y(t) = t^2 + \frac{1}{2}\int_0^t y(v)\sin(t-v)\,dv$. (3)

(b) Use Euler-Lagrange condition to show that the shortest distance between two points in a plane is a straight line.